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Gevrey Hypoellipticity for Extended Grushin Class and FBI-Transformation

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§1. Introduction

In this monograph we shall mention only the main results obtained recently on Gevrey hypoellipticity for extended class of Grushin operators. Precise proof of them will be given in a forthcoming paper. We shall determine the non-isotropic Gevrey exponents for Grushin operators by using also the method of FBI-transformation given in [1] somewhat modifying it as well as by using the method of pseudodifferential operators. Thus, we get an amelioration of the results obtained in the previous papers [3] and [4].

§2. Gevrey functions and FBI-transformation

We denote $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $D_j = -i\partial_{x_j}$, $j = 1, \dots, n$, as usual. We remember the definition of Gevrey functions.

Definition 2.1. Let Ω be an open set in \mathbb{R}^n and $\phi \in C^\infty(\Omega)$. Then we say that $\phi \in G^{\{s\}}(\Omega)$, $s = (s_1, \dots, s_n)$, $s_j > 0$, if for any compact subset K of Ω there are positive constants C_0 and C_1 such that

$$(2.1) \quad \sup_{x \in K} |D^\alpha \phi(x)| \leq C_0 C_1^{|\alpha|} |\alpha|^{\langle s, \alpha \rangle}, \quad \alpha \in \mathbb{Z}_+^n,$$

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where $\langle s, \alpha \rangle = s_1 \alpha_1 + \dots + s_n \alpha_n$.

Proposition 2.1. *Let $\varphi \in C_0^\infty(\Omega)$. If for any compact subset K of Ω there are positive constants C_0 and C_1 such that*

$$\sup_{x \in K} |D_{x_j}^k \varphi(x)| \leq C_0 C_1^k k!^{s_j}, \quad j = 1, 2, \dots, n, \quad k \in Z_+.$$

Then we have $\varphi \in G^{\{s_1, s_2, \dots, s_n\}}(\Omega)$.

The proof can be obtained by using FBI-transformation whose definition will be given in (2.2).

Proposition 2.2. *Let a be a positive parameter. For any $\varepsilon, 0 < \varepsilon < 1$, there exists a positive constant C_ε such that*

$$|\partial_x^k e^{-ax^2}| \leq C_\varepsilon^{k+1} a^{\frac{k}{2}} k!^{\frac{1}{2}} e^{-\varepsilon ax^2}, \quad -\infty < x < \infty, \quad k = 1, 2, \dots$$

Let $u(x) \in C_0^\infty(\mathbf{R}^n)$. Then we have the Fourier inversion formula

$$u(x) = (2\pi)^{-n} \iint_{\mathbf{R}^n \times \mathbf{R}^n} u(y) e^{i\langle x-y, \xi \rangle} dy d\xi.$$

Now shift the contour of integration from $\mathbf{R}^n \times \mathbf{R}^n \subset \mathbf{R}^n \times \mathbf{C}^n$ to the contour

$$\Gamma(y, \xi) = (y, \xi + i\langle \xi \rangle(x - y)), \quad (y, \xi) \in \mathbf{R}^n \times \mathbf{R}^n.$$

Then we have the formula

$$u(x) = (2\pi)^{-n} \iint_{\mathbf{R}^n \times \mathbf{R}^n} u(y) e^{i\langle x-y, \xi \rangle - \langle \xi \rangle (x-y)^2} \alpha(x - y, \xi) dy d\xi,$$

where

$$\langle \xi \rangle = (1 + \xi^2)^{\frac{1}{2}} = (1 + \xi_1^2 + \dots + \xi_n^2)^{\frac{1}{2}}$$

and

$$\alpha(x - y, \xi) = \prod_{j=1}^n (1 + i(x_j - y_j)\xi_j (1 + \xi^2)^{-\frac{1}{2}}).$$

From this formula we define the FBI-transformation of $u(x)$ by

$$(2.2) \quad \mathcal{F}u(x, \xi) = \int_{\mathbf{R}^n} u(y) e^{i\langle x-y, \xi \rangle - \langle \xi \rangle (x-y)^2} \alpha(x - y, \xi) dy.$$

The result of M. Christ, [1] is modified slightly in the following theorem relating to the characterization of the class $G^{\{s\}}$.

Theorem 2.3. (cf. [1], Theorem 2.3.) *Let $s = (s_1, s_2, \dots, s_n)$, $s_j \geq 1$, $j = 1, 2, \dots, n$, and $u(x) \in C_0^\infty(\mathbf{R}^n)$. Then the following four assertions are mutually equivalent:*

- (a) $u(x) \in G^{\{s\}}$ in a neighborhood of $x_0 \in \mathbf{R}^n$.
- (b) There exist $C, \delta \in \mathbf{R}_+$ and a neighborhood V of x_0 such that

$$|\mathcal{F}u(x, \xi)| \leq C e^{-\delta \sum_{j=1}^n |\xi_j|^{\frac{1}{s_j}}}, \quad (x, \xi) \in V \times \mathbf{R}^n.$$

- (c) There exist an open neighborhood $U = U(x_0) \subset \mathbf{C}^n$ of x_0 and $C, \delta \in \mathbf{R}_+$ such that, for each $\lambda \in \mathbf{R}_+^n$, $|\lambda| \geq 1$, there exists a decomposition

$$u = g_\lambda + h_\lambda \quad \text{in } U \cap \mathbf{R}^n$$

such that g_λ is holomorphic in U ,

$$|g_\lambda(z)| \leq C e^{C|\lambda||\operatorname{Im}(z)|}, \quad z \in U$$

and

$$|h_\lambda(x)| \leq C e^{-\delta \sum_{j=1}^n \lambda_j^{\frac{1}{s_j}}}, \quad x \in U \cap \mathbf{R}^n.$$

- (d) There exist an open neighborhood $U = U(x_0) \subset \mathbf{C}^n$ of x_0 and $C, \delta \in \mathbf{R}_+$ such that for each $\lambda \in \mathbf{R}_+^n$, $|\lambda| \geq 1$, there exists a decomposition

$$u = g_\lambda + h_\lambda \quad \text{in } U \cap \mathbf{R}^n$$

such that g_λ is holomorphic in $\{z \in U; |\operatorname{Im}(z_j)| \leq \langle \lambda \rangle_s |\lambda|^{-1}\} \equiv U_\lambda$,

$$|g_\lambda(z)| \leq C, \quad z \in U_\lambda,$$

and

$$|h_\lambda(x)| \leq C e^{-\delta \sum_{j=1}^n \lambda_j^{\frac{1}{s_j}}}, \quad x \in U \cap \mathbf{R}^n.$$

Remark 2.1. By using appropriate cut-off functions for u , the standard method of calculation goes well to prove (a) \iff (b) with the aid of Proposition 2.2. Proof that (b) \implies (c) \implies (d) \implies (b) can be obtained by the

same method as in [1]. However, it might be needed to add a sketch of the proof of (a) \implies (b). it will be sufficient to consider one dimensional case.

(i) The case where $s = 1$. Let $u \in C_0^\infty(\mathbf{R})$ and let u be real analytic in a neighborhood of x_0 , say in $\omega_\delta = \{x; |x - x_0| < \delta\}$ for some $\delta > 0$. Then for

$$\mathcal{F}u(x, \xi) = \int u(y) e^{i(x-y, \xi) - \langle \xi \rangle (x-y)^2} \alpha(x-y, \xi) dy$$

we make a deformation of the integral contour in ω_δ and we have

$$|\mathcal{F}u(x, \xi)| \leq C e^{-c\langle \xi \rangle}, \quad (x, \xi) \in V \times \mathbf{R},$$

where V is a small neighborhood of x_0 and C and c are positive constants independent of ξ .

(ii) The case where $s > 1$. We may suppose that $u \in C_0^\infty(\omega_\delta) \cap G^{\{s\}}$, so that we have $u(y)\alpha(x-y, \xi) \in C_0^\infty(\omega_\delta) \cap G^{\{s\}}$. By Proposition 2.2, taking C_1, C, C' sufficiently large and c' sufficiently small we have

$$\begin{aligned} & \left| \xi^{-N} \int e^{i(x-y)\xi} D_y^N (u(y)\alpha e^{-\langle \xi \rangle (x-y)^2}) dy \right| \\ & \leq |\xi|^{-N} C_1^{N+1} \sum_{j=0}^N \binom{N}{j}^s j!^{\frac{1}{2}} \langle \xi \rangle^{\frac{j}{2}} (N-j)!^s \\ & \leq |\xi|^{-N} C^{N+1} N!^s \frac{\langle \xi \rangle^{\frac{j}{2}}}{j!^{s-\frac{1}{2}}} \\ & \leq \left(\frac{CN^s}{|\xi|} \right)^N e^{(s-\frac{1}{2})|\xi|^{\frac{1}{2s-1}}}. \end{aligned}$$

Now take N such that $|N - (\varepsilon|\xi|)^{\frac{1}{s}}| < 1$ with ε sufficiently small. Then the above quantity is estimated by

$$C(C\varepsilon)^N e^{(s-\frac{1}{2})|\xi|^{\frac{1}{2s-1}}} \leq C' e^{-c'|\xi|^{\frac{1}{2}}}, \quad x \in \omega_{\frac{\delta}{2}}, \xi \in \mathbf{R}. \quad \square$$

Remark 2.2. In Theorem 2.3, we can replace $\mathcal{F}u(x, \xi)$ by $\mathcal{F}_s u(x, \xi)$ as follows:

$$(2.3) \quad \mathcal{F}_s u(x, \xi) = \int u(y) e^{i(x-y, \xi) - \langle \xi \rangle_s (x-y)^2} \alpha_s(x-y, \xi) dy$$

$$\langle \xi \rangle_s = \sum_{j=1}^n (1 + \xi_j^2)^{\frac{1}{2s_j}},$$

$$\alpha_s(x - y, \xi) = \prod_{j=1}^n \left(1 + \frac{i}{s_j} (x_j - y_j) \xi_j (1 + \xi_j^2)^{\frac{1}{2s_j} - 1}\right).$$

The transform $\mathcal{F}u(z, \xi)$ extends, for each ξ , to an entire holomorphic function of $z \in \mathbb{C}^n$. We can see that the same reasoning as in (c) and (d) gives

$$(2.4) \quad |\mathcal{F}u(z, \xi)| \leq C e^{-\delta \sum_{j=1}^n |\xi|^{\frac{1}{s_j}}} e^{C|\xi||\operatorname{Im}(z)|}$$

for z in a sufficiently small neighborhood $V \subset \mathbb{C}^n$ of x_0 .

§3. Main results

We shall give the definition of the extended class of Grushin operators. We write $(x, y) = (x_1, \dots, x_k, y_1, \dots, y_n) \in \mathbb{R}^{k+n}$. Let m be an even positive integer and let $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k)$, $q = (q_1, q_2, \dots, q_k)$ whose elements are rational numbers such that

$$\sigma_1, \dots, \sigma_p > 0, \sigma_{p+1} = \dots = \sigma_k = 0, (0 \leq p \leq k)$$

$$q_1 \geq q_2 \geq \dots \geq q_p \geq 0, q_{p+1} \geq \dots \geq 0, \quad q_1 > 0.$$

Furthermore, we assume

$$mq_j \in \mathbb{Z}, j = 1, \dots, k; \quad \frac{mq_j}{\sigma_j} \in \mathbb{Z}, j = 1, 2, \dots, p.$$

We pose the following major hypothesis:

Hypothesis (G) We suppose $1 + q_p > \sigma_0 = \max(\sigma_1, \dots, \sigma_p)$.

Remark 3.1. Grushin's original major hypothesis given in [2] was $1 + q_k > \sigma_0 = \max(\sigma_1, \dots, \sigma_p)$. We shall see that we can weaken this condition as above. (See §5 and §6.) The assumption on q_1, \dots, q_k given in [4] is also slightly weakened as above. When $p = 0$, we consider $q_0 = 0, \sigma_0 = 0$.

We divide x into two parts such as $x = (x', x'')$ when $1 \leq p < k$, where $x' = (x_1, \dots, x_p)$ and $x'' = (x_{p+1}, \dots, x_k)$. We consider $x = x'$ when $p = k$ and $x = x''$ when $\sigma = (0, \dots, 0)$. Now we shall consider a differential operator with polynomial coefficients under the hypothesis (M):

$$(3.1) \quad P(x', y, D_x, D_y) = \sum_{\substack{\langle \sigma, \nu \rangle + |\gamma| = \langle q, \alpha \rangle + |\alpha + \beta| - m \\ |\alpha + \beta| \leq m}} a_{\alpha\beta\nu\gamma} x'^\nu y^\gamma D_x^\alpha D_y^\beta, \quad a_{\alpha\beta\nu\gamma} \in \mathbb{C},$$

$$\alpha, \nu \in \mathbb{Z}_+^k, \quad \beta, \gamma \in \mathbb{Z}_+^n,$$

where $a_{\alpha\beta\nu\gamma}$ can be non zero only when $|\gamma| = \langle q, \alpha \rangle + |\alpha + \beta| - m - \langle \sigma, \nu \rangle$ is a non negative integer and we write such as $|\alpha + \beta| = |\alpha| + |\beta|$. We may also consider $\nu = (\nu_1, \dots, \nu_p, 0, \dots, 0)$.

We can see the symbol $P(x', y, \xi, \eta)$ satisfies the following condition.

Condition 1. (quasi-homogeneity) We have

$$P(\lambda^{-\sigma} x', \lambda^{-1} y, \lambda^{1+q} \xi, \lambda \eta) = \lambda^m P(x', y, \xi, \eta), \quad \lambda > 0, x, \xi \in \mathbb{R}^k, y, \eta \in \mathbb{R}^n,$$

where $\lambda^{-\sigma} x' = (\lambda^{-\sigma_1} x_1, \dots, \lambda^{-\sigma_p} x_p)$ and $\lambda^{1+q} \xi = (\lambda^{1+q_1} \xi_1, \dots, \lambda^{1+q_k} \xi_k)$.

We add the two more conditions on P .

Condition 2. (ellipticity) The operator P is elliptic for $|x'| + |y| = 1$.

Condition 3. (non-zero eigenvalue) For all $\omega, |\omega| = 1$, the equation

$$P(x', y, \omega, D_y) v(y) = 0 \quad \text{in } \mathbb{R}_y^n$$

has no non-trivial solution in $\mathcal{S}(\mathbb{R}_y^n)$.

We set the Gevrey indices as follows.

$$\theta_j = \max\left(\frac{1 + q_j}{1 + q_k}, \frac{1 + q_p}{1 + q_p - \sigma_0}\right) \quad \text{for } j = 1, \dots, p,$$

$$\theta_j = \frac{1 + q_j}{1 + q_k} \quad \text{for } j = p + 1, \dots, k, \quad d = \max_{1 \leq j \leq k} \left\{ \frac{\theta_j + q_j}{1 + q_j} \right\},$$

We also denote

$$d = \max_{1 \leq j \leq k} \left\{ \frac{\theta_j + q_j}{1 + q_j} \right\} \cdot I_n = (d, \dots, d).$$

Theorem 3.2. (cf. [4]) *Let Ω be an open neighborhood of $(0, 0)$ and consider the equation*

$$(3.2) \quad P(x', yD_x, D_y)u(x, y) = f(x, y) \quad \text{in } \Omega,$$

where $u(x, y) \in \mathcal{D}'(\Omega)$ and $f(x, y) \in G_{x,y}^{\{\theta,d\}}(\Omega)$. Then we have $G_{x,y}^{\{\theta,d\}}(\Omega)$.

Remark 3.3. In the above theorem we can see that

$$(i) \quad p = 0, \theta_1 = 1 \iff (\theta, d) = (1, \dots, 1),$$

$$(ii) \quad p = 0, \theta_1 > 1 \implies 1 < d = \frac{\theta_1 + q_1}{1 + q_1} < \theta_1.$$

Examples (a) For the operator $P_1 = D_y^2 + y^{2k}D_x^2$, ($k = 1, 2, \dots$, $p = 0, q_1 = k, \sigma_1 = 0$ and $\theta_1 = 1, d = 1$).

(b) For the operator $P_2 = D_y^2 + (x^{2l} + y^{2k})D_x^2$, ($k, l = 1, 2, \dots$),

$$q_1 = k, \sigma_1 = k/l \text{ and } \theta_1 = \frac{l(1+k)}{l(1+k)-k}, d = \frac{\theta_1 + k}{1+k}.$$

(c) For the operator $P_3 = D_y^2 + (x^{2l} + y^{2k})(D_x^2 + D_z^2)$, ($k, l = 1$, have

$$q_1 = q_2 = k, \sigma_1 = k/l, \sigma_2 = 0, x' = x, x'' = z; \theta_1 = \frac{l(1+k)}{l(1+k)-k}, \theta_2 =$$

(d) For the operator $P_4 = D_y^2 + (x^{2l} + y^{2k})D_x^2 + D_z^2$, ($k, l = 1$, have

$$q_1 = k, q_2 = 0, \sigma_1 = k/l, \sigma_2 = 0; \theta_1 = 1 + k, \theta_2 = 1, d = \frac{\theta_1 + k}{1+k} =$$

We remark that this operator P_4 does not satisfy the original hypo of Grushin.

(e) An example with $1 < d_1 < d_2$ is given by $P_4 = D_y^2 + (x^4 + y^4)D_x^2 + (x^2 + y^2)D_z^2$, where we have

$$q_1 = 2, q_2 = 1, \sigma_1 = \sigma_2 = 1; \theta_1 = \theta_2 = 2, d_1 = \frac{4}{3} < d_2 = \frac{3}{2}, d = \frac{3}{2}.$$

Remark 3.4. We omit the proof of C^∞ -hypoellipticity of the operator P given in Theorem 3.2 since it is much simpler than that of Gevrey hypoellipticity. Then by using a cut-off function for u , we may suppose that $u, f \in C_0^\infty(\Omega)$ and $f \in G_{x,y}^{\{\theta,d\}}$ in a neighborhood of $(0,0) \in \mathbf{R}_{x,y}^{k+n}$. By Theorem 2.3,(b), our main purpose becomes to prove that there exist a small neighborhood V of $(0,0)$ and positive constants C and δ such that

$$(3.3) \quad |\mathcal{F}u(\tilde{x}, \tilde{y}, \xi, \eta)| \leq e^{-\delta(\sum_{j=1}^k |\xi_j|^{\frac{1}{\theta_j}} + |\eta|^{\frac{1}{d}})}, \quad (\tilde{x}, \tilde{y}, \xi, \eta) \in V \times \mathbf{R}_{\xi,\eta}^{k+n},$$

where

$$\begin{aligned} \mathcal{F}(\tilde{x}, \tilde{y}, \xi, \eta) &= \int u(x, y) e^{i(\langle \tilde{x}-x, \xi \rangle + \langle \tilde{y}-y, \eta \rangle) - \langle \mu \rangle ((\tilde{x}-x)^2 + (\tilde{y}-y)^2)} \alpha(\tilde{x} - x, \xi) \\ &\quad \cdot \alpha(\tilde{y} - y, \eta) dx dy, \quad \mu = (\xi, \eta), \\ \alpha(\tilde{x} - x, \xi) &= \prod_{j=1}^k (1 + i)(\tilde{x}_j - x_j) \xi_j (1 + \xi^2)^{-\frac{1}{2}}, \\ \alpha(\tilde{y} - y, \eta) &= \prod_{j=1}^n (1 + i)(\tilde{y}_j - y_j) \eta_j (1 + \eta^2)^{-\frac{1}{2}}. \end{aligned}$$

We can prove the inequality (3.3) in three steps. We prove first the inequality (3.3) in the elliptic region:

$$R_E = \{(\xi, \eta); (\xi, \eta) \in \mathbf{R}_{\xi,\eta}^{k+n}, |\xi| \leq |\eta|\}.$$

Next, we prove the inequality (3.3) in the subelliptic region:

$$R_S = \{(\xi, \eta); (\xi, \eta) \in \mathbf{R}_{\xi,\eta}^{k+n}, (\frac{1}{c} \sum_{j=1}^k |\xi_j|^{\frac{1}{\theta_j}})^d \leq |\eta| \leq |\xi|\}, \quad c > 0.$$

Finally, we obtain the inequality of the kind (3.3) in the L^2 -sense in the degenerate region:

$$R_D = \{(\xi, \eta); (\xi, \eta) \in \mathbf{R}_{\xi,\eta}^{k+n}, c|\eta|^{\frac{1}{d}} \leq \sum_{j=1}^k |\xi_j|^{\frac{1}{\theta_j}}\}, \quad c > 0.$$

These steps will be completed by a precision of the method given in [1] and [4].

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